3.2 – Norm, Dot Product, and Distance in R^n

Definition: If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a vector in R^n, then the **norm** of $\mathbf{v}$ (also called its **length** or **magnitude**) is denoted by $\|\mathbf{v}\|$ [by this author], and is defined by the formula $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.

Theorem 3.2.1 Properties of the norm of a vector
If $\mathbf{v}$ is a vector in R^n and k is any scalar, then:
a) $\|\mathbf{v}\| \geq 0$
b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
c) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$

Definition: The norm of a **unit vector** is 1. We can obtain a unit vector from a nonzero vector $\mathbf{v}$ by multiplying by the reciprocal of its length. This process is called **normalizing** the vector.

#1 Find the norm of $\mathbf{v}$ and a unit vector that is oppositely directed to $\mathbf{v}$.
a. $\mathbf{v} = (2,\ 2,\ 2)$
b. $\mathbf{v} = (1,\ 0,\ 2,\ 1,\ 3)$

Definition: The **standard unit vectors in $\mathbf{R}^n$** are the standard basis vectors for R^n, $\mathbf{e}_1 = (1,\ 0,\ 0,\ \dots,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0,\ \dots,\ 0)$, $\dots$, $\mathbf{e}_n = (0,\ 0,\ 0,\ \dots,\ 1)$.

Definition: If $\mathbf{u} = (u_1, u_2, \ldots,\ u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots,\ v_n)$ are points in R^n, then we denote the **distance** between **u** and **v** by $d(\mathbf{u},\ \mathbf{v})$ and define it to be
$d\,(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$

#4 Evaluate the given expression with $\mathbf{u} = (2,\ -2,\ 3)$, $\mathbf{v} = (1,\ -3,\ 4)$, and $\mathbf{w} = (3,\ 6,\ -4)$.

a. $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$

b. $\|\mathbf{u} - \mathbf{v}\|$

c. $\|3\mathbf{v}\| - 3\,\|\mathbf{v}\|$

d. $\|\mathbf{u}\| - \|\mathbf{v}\|$

Definition: Let **u** and **v** be nonzero vectors in R^n positioned so that their initial points coincide. The **angle between u and v** is the angle θ determined by **u** and **v** such that $0 \leq \theta \leq \pi$.

Definition: If **u** and **v** are nonzero vectors in R^2 or R^3, and if θ is the angle between **u** and **v**, then the **dot product** or **Euclidean inner product** of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

Definition: If $\mathbf{u} = (u_1, u_2, \ldots, \ u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, \ v_n)$ are vectors in R^n, then the **dot product** or **Euclidean inner product** of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.

#10 Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, and $\mathbf{v} \cdot \mathbf{v}$.

a. $\mathbf{u} = (1,\ 1,\ -2,\ 3)$, $\mathbf{v} = (-1,\ 0,\ 5,\ 1)$

b. $\mathbf{u} = (2,\ -1,\ 1,\ 0,\ -2)$, $\mathbf{v} = (1,\ 2,\ 2,\ 2,\ 1)$

#11 Find the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$ and the cosine of the angle between those vectors. State whether that angle is acute, obtuse, or 90°.

a. $\mathbf{u} = (3,\ 3,\ 3)$, $\mathbf{v} = (1,\ 0,\ 4)$

b. $\mathbf{u} = (0,\ -2,\ -1,\ 1)$, $\mathbf{v} = (-3,\ 2,\ 4,\ 4)$

Theorem 3.2.2 Properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and if k is a scalar, then:

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry property)

b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive property)

c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ (homogeneity property)

d) $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (positivity property)

Theorem 3.2.3 More properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and if k is a scalar, then:

a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Theorem 3.2.4 Cauchy-Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \dots u_n)$ and $\mathbf{v} = (v_1, v_2, \dots v_n)$ are vectors in R^n, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ or in terms of components

$|u_1 v_1 + u_2 v_2 + \dots + u_n v_n| \leq \left(u_1^2 + u_2^2 + \dots u_n^2\right)^{\frac{1}{2}} \left(v_1^2 + v_2^2 + \dots v_n^2\right)^{\frac{1}{2}}$.

Theorem 3.2.5 Triangle Inequalities

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, then:

a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality for vectors)

b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (triangle inequality for distances)